Solvable multi-species extensions of the drop-push model

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Abstract. A family of multispecies drop-push systems on a one-dimensional lattice is investigated. It is shown that this family is solvable in the sense of the Bethe ansatz, provided a nonspectral matrix equation is satisfied. The large-time behavior of the conditional probabilities, and the dynamics of the particle-type change are also investigated.

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1 Introduction

Various aspects of one-dimensional asymmetric exclusion processes have been of physical interest. These contain, for example, the kinetics of biopolymerization [1], dynamical models of interface growth [2], and the traffic models [3]. This model is also related to the noisy Burgers equation [4], and hence to the study of shocks [5,6]. The dynamical properties of this model have also been extensively studied, for example in [6–8].

In the study of stochastic processes, the term solvable has been used in several meanings. In [9–16], solvability means solvability in the sense of the Bethe ansatz, or factorization of the N-particle scattering matrix to the two-particle scattering matrices. This is related to the fact that for systems solvable in this sense, there are a large number of conserved quantities. In [17–26], solvability means closedness of the evolution equation of the empty intervals (or their generalization). And in [27–29], solvability means that the evolution equation for the n-point functions, contain only n- or less- point functions. Models discussed in [27], are also solvable in the sense of the Bethe ansatz.

In [9], the Bethe ansatz was used to solve the asymmetric simple exclusion process on a one-dimensional lattice. In [10], a similar technique was used to solve the drop-push model [30], and a one-parameter family of reactions containing the simple exclusion processes and the drop-push model as special cases. In [12], the same technique was used to solve a two-parameter family of processes, involving bidirectional diffusion, exclusion, and pushing. The behavior of this last model on the continuum, was investigated in [11]. In all of the above cases, the essence

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of the method has been to replace the reactions by suitable boundary conditions. This is very much like what is done, for example, in studying the scattering of particles from hard spheres, where instead of the reaction one uses the boundary condition that the wave function vanishes at the surface of the sphere. The boundary conditions involved were extended in [15] to describe systems in them the process of annihilation exists as well, and in [16], to general boundary conditions in the continuum.

All of the above studies have been about single-species systems. In [13,14], systems with exclusion processes were investigated, which contained more than one species. It was shown in [13], that in order that such systems be solvable in the sense of the Bethe ansatz, certain relations should be satisfied between the rates. This relations can be written as some kind of a spectral Yang-Baxter equation. In [14], it was shown that this spectral equation is equivalent to a nonspectral matrix equation involving the rates.

Here we want to extend this approach to the case of drop-push models. The systems under consideration, consist of N particles, which can be of several species. This particles live on an infinite one-dimensional lattice, so that each site of the lattice is either empty or contains one particle. Each particle hops to the site at its right-hand side with unit rate (this is just a scaling of the time), if that site is empty. If that site is occupied, then the particle may push the other particle, and at the same time a reaction may occur between these neighboring particles changing their species. As an example, one can consider a flow of gas moving on the surface of a catalyst. Some of the gas molecules attach to the surface sites. They can hop to adjacent sites, if these sites are empty, and two molecules in adjacent sites may interact, while one pushes the other. Another example is the car traffic. The reaction rates of

these systems do not necessarily satisfy the solvability conditions. However, studying solvable systems also permits one to shed light on a system the reaction rates of which are in a neighborhood of those of a solvable system, for example using perturbation techniques.

The scheme of the paper is the following. In Section 2, a multi-species extension of the drop-push model is introduced and the use of a suitable boundary condition instead of the reaction is investigated. In Section 3, the solvability condition (in the sense of the Bethe-ansatz) for this reaction is investigated, and it is shown that this condition, which is a spectral equation for the matrix of the reaction rates, can be rewritten as a nonspectral equation for the same matrix. In Section 4, the conditional probability is obtained, and its behavior in the two-particle sector, specially its large-time behavior, is investigated. In Section 5, the dynamics of the particle-type change in the two-particle sector is investigated and the large-time limit of the probability of particle-types is obtained. Finally, Section 6 is devoted to the concluding remarks.

2 Multi-species extension of the drop-push model

In the ordinary drop-push model, the system consists of a single type of particles, living on a one-dimensional lattice. Each site of the lattice is either empty or occupied by one particle. Any particle can hop to the site at its right neighbor, with the rate 1, if that site is empty. If the right neighbor site is occupied, the particle can still hop to that site and push the second particle, with the same rate 1. One can write the reactions like

$$A \underbrace{A \cdots A}_{n} \emptyset \to \emptyset A \underbrace{A \cdots A}_{n}, \qquad \text{with the rate 1.} \quad (1)$$
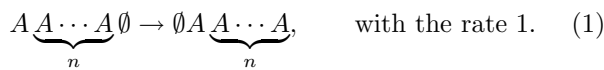
In this reaction, the total number of particles is conserved. For a system containing N particles, the question of interest is to determine the probability of finding the N particles in sites x_1 to x_N, where

$$x_i < x_j, \qquad \text{for } i < j, \quad (2)$$

the so called physical region. It is easily seen that the evolution equation for this probability, $P(x_1, \ldots, x_N; t)$, is

$$\dot{P}(x_1, \ldots, x_N; t) = P(x_1 - 1, \ldots, x_N; t)$$
$$+ \cdots + P(x_1, \ldots, x_N - 1; t) - N P(x_1, \ldots, x_N; t), \quad (3)$$

if among the sites x_i, no two are adjacent; that is, if $x_i < x_{i+1} - 1$. For a block of $(n+1)$ adjacent sites, the evolution
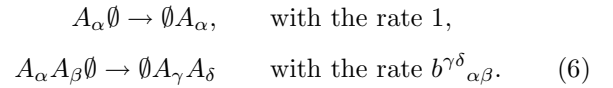
equation becomes

$$\dot{P}(x_0 = x, \ldots, x_n = x + n; t) =$$
$$P(x_0 = x - 1, \ldots, x_n = x + n; t) + \cdots$$
$$+ P(x_0 = x - 1, \ldots, x_k = x + k - 1,$$
$$x_{k+1} = x + k + 1, \ldots, x_n = x + n; t)$$
$$+ \cdots - (n + 1)P(x_0 \ldots, x_n; t). \quad (4)$$

This looks different from (3). However, defining a boundary condition

$$P(\ldots, x, x, \ldots) := P(\ldots, x - 1, x, \ldots), \quad (5)$$

makes the forms of (4) and (3) similar. One notes that $P(\ldots, x, x, \ldots)$ has in fact no physical meaning, since the argument of P in that expression is not in the physical region. But its introduction helps to solve the evolution equation, as it was done in [26] (and in [9] for the exclusion process).

Now suppose that the system consists of k species of particles; that is, an occupied site may have k different states. Assume moreover, that if the right neighbor of a particle is free, the reaction is the same as ordinary drop-push model, without changing the type of the particle, but there is a difference when two particles are adjacent to each other: the left particle does push the right one with unit rate, but in the mean time there is a probability that the types of the particles change. So we have reactions like

$$A_\alpha \emptyset \to \emptyset A_\alpha, \qquad \text{with the rate 1,}$$
$$A_\alpha A_\beta \emptyset \to \emptyset A_\gamma A_\delta \qquad \text{with the rate } b^{\gamma\delta}{}_{\alpha\beta}. \quad (6)$$

Consider a system consisting of two particles, and denote the probability that the first particle be at the site x and of the type A_α and the second particle be at the site y and of the type A_β, by $P^{\alpha\beta}(x, y)$. The evolution equations become

$$\dot{P}^{\alpha\beta}(x, y; t) = P^{\alpha\beta}(x - 1, y; t) + P^{\alpha\beta}(x, y - 1; t)$$
$$- 2P^{\alpha\beta}(x, y; t), \quad x < y - 1,$$
$$\dot{P}^{\alpha\beta}(x, x + 1; t) = P^{\alpha\beta}(x - 1, x + 1; t)$$
$$+ b^{\alpha\beta}{}_{\gamma\delta} P^{\gamma\delta}(x - 1, x; t) - B^{\alpha\beta}$$
$$\times P^{\alpha\beta}(x, x + 1; t) - P^{\alpha\beta}(x, x + 1; t), \quad (7)$$

where

$$B^{\alpha\beta} := \sum_{\gamma\delta} b^{\gamma\delta}{}_{\alpha\beta}. \quad (8)$$

In fact, $B^{\alpha\beta}$ is the overall pushing rate, in which the type change is unimportant. If this overall pushing rate is 1 (that is equal to the hoping rate), the second equation in (7) is simplified and it is seen that it can be rewritten in the form of the first equation, provided one introduces the boundary condition

$$P^{\alpha\beta}(x, x; t) = b^{\alpha\beta}{}_{\gamma\delta} P^{\gamma\delta}(x - 1, x; t). \quad (9)$$

One notes that all of the elements of the matrix b (including the diagonal elements) are nonnegative, as they are rates, and b satisfies

$$(\mathbf{s} \otimes \mathbf{s})b = \mathbf{s} \otimes \mathbf{s}, \tag{10}$$

where

$$s_\alpha := 1. \tag{11}$$

(This simply means that the sum of the elements each of the columns of b is equal to one.) A similar matrix b was introduced in [14], however the diagonal elements of that b were not necessarily nonnegative.

Now consider a system consisting of N particles of various species, with the evolution equation

$$\dot{\mathbf{P}}(x_1, \ldots, x_N; t) =$$
$$\mathbf{P}(x_1 - 1, \ldots, x_N; t) + \cdots + \mathbf{P}(x_1, \ldots, x_N - 1; t)$$
$$- N\,\mathbf{P}(x_1, \ldots, x_N; t), \tag{12}$$

in the whole physical region, and the boundary condition

$$\mathbf{P}(\ldots, x_k = x, x_{k+1} = x, \ldots) :=$$
$$b_{k,k+1}\,\mathbf{P}(\ldots, x_k = x - 1, x_{k+1} = x, \ldots), \tag{13}$$

where

$$b_{k,k+1} := 1 \otimes \cdots \otimes 1 \otimes \underbrace{b}_{k,k+1} \otimes 1 \otimes \cdots \otimes 1, \tag{14}$$

and $\mathbf{P}$ is an N-tensor the components of which are probabilities. It is seen that in this system, apart from the simple diffusion, there is a reaction between a block of $n + 1$ adjacent particles:

$$A_{\alpha_0} \cdots A_{\alpha_n} \emptyset \to \emptyset A_{\gamma_0} \cdots A_{\gamma_n},$$
$$\text{with the rate} \quad (b_{n-1,n} \cdots b_{0,1})^{\gamma_0 \cdots \gamma_n}{}_{\alpha_0 \cdots \alpha_n}. \tag{15}$$

This comes from the fact that

$$\mathbf{P}(x_0 = x, \ldots, x_{n-1} = x + n - 1, x_n = x + n - 1) =$$
$$(b_{n-1,n} \cdots b_{0,1})$$
$$\times \mathbf{P}(x_0 = x - 1, \ldots, x_{n-1} = x + n - 2, x_n = x + n - 1). \tag{16}$$

Note the order of the matrices b. This order suggests that if a collection of $n + 1$ particles are adjacent, there is a probability that the first particle pushes the second and changes the type of the second (and itself) and then it is the second (modified) particle that interacts with the third.

3 Solvability and the Bethe-ansatz solution

Consider the evolution equation (12) with the boundary condition (13). To solve this equation, one as usual seeks the eigenvectors of the operator acting at the right-hand side of (12), that is, one tries to solve

$$E\,\mathbf{\Psi}(x_1, \ldots, x_N) =$$
$$\mathbf{\Psi}(x_1 - 1, \ldots, x_N) + \cdots + \mathbf{\Psi}(x_1, \ldots, x_N - 1)$$
$$- N\,\mathbf{\Psi}(x_1, \ldots, x_N), \tag{17}$$

with

$$\mathbf{\Psi}(\ldots, x_k = x, x_{k+1} = x, \ldots) :=$$
$$b_{k,k+1}\,\mathbf{\Psi}(\ldots, x_k = x - 1, x_{k+1} = x, \ldots). \tag{18}$$

The Bethe-ansatz solution to this equation is

$$\mathbf{\Psi}(\vec{x}) = \sum_\sigma A_\sigma\, e^{i\sigma(\vec{p}) \cdot \vec{x}}\, \mathbf{\Xi}, \tag{19}$$

where $\mathbf{\Xi}$ is an arbitrary vector and the summation runs over the elements of the permutation group of N objects. Putting this in (17), one arrives at

$$E = \sum_{k=1}^N (e^{-ip_k} - 1), \tag{20}$$

while (18) gives

$$[1 - e^{-i\sigma(p_k)}\,b_{k,k+1}]A_\sigma + [1 - e^{-i\sigma(p_{k+1})}\,b_{k,k+1}]A_{\sigma\sigma_k} = 0, \tag{21}$$

where

$$\sigma_k(p_j) = \begin{cases} p_{k+1}, & j = k \\ p_k, & j = k + 1 \\ p_j, & j \neq k, k + 1. \end{cases} \tag{22}$$

From (21), one arrives at

$$A_{\sigma\sigma_k} = S_{k,k+1}[\sigma(p_k), \sigma(p_{k+1})]A_\sigma, \tag{23}$$

where

$$S_{k,l}(p_i, p_j) := -(1 - z_j\,b_{k,l})^{-1}(1 - z_i\,b_{k,l}), \tag{24}$$

and

$$z_j := e^{-ip_j}. \tag{25}$$

So one can construct A_σ's from A_1, by writing σ as a product of σ_k's; that is, one can write the N-particle scattering matrix A, as a product of the two-particle scattering matrices S. However, as the generators of the permutation group satisfy

$$\sigma_k\sigma_{k+1}\sigma_k = \sigma_{k+1}\sigma_k\sigma_{k+1}, \tag{26}$$

one also needs

$$A_{\sigma_k\sigma_{k+1}\sigma_k} = A_{\sigma_{k+1}\sigma_k\sigma_{k+1}}. \tag{27}$$

This, in terms of the S becomes

$$S_{1,2}(p_2, p_3)\,S_{2,3}(p_1, p_3)\,S_{1,2}(p_1, p_2) =$$
$$S_{2,3}(p_1, p_2)\,S_{1,2}(p_1, p_3)\,S_{2,3}(p_2, p_3). \tag{28}$$

In terms of the R-matrix defined through

$$S_{k,k+1} =: \Pi_{k,k+1}\, R_{k,k+1}, \tag{29}$$

(28) becomes

$$R_{2,3}(p_2,p_3)\, R_{1,3}(p_1,p_3)\, R_{1,2}(p_1,p_2) =$$
$$R_{1,2}(p_1,p_2)\, R_{1,3}(p_1,p_3)\, R_{2,3}(p_2,p_3), \tag{30}$$

which is the spectral Yang-Baxter equation.

The Bethe-ansatz solution exists, iff the scattering matrix satisfies (28). This is a general statement not coming from a specific reaction. In the problem studied here, S is of the form (24). Comparing this with the S-matrix obtained for the multi-species simple exclusion process [14], it is seen that $S(p_i,p_j)$ in these two different problems are transformed to each other by a simple change $z_i \leftrightarrow z_j$. The definition of z in terms of p in the present paper is, however, different from that of [14]. It is also seen that with the changes $b_{1,2} \leftrightarrow b_{2,3}$ and $z_1 \leftrightarrow z_3$, (28) is transformed to the spectral equation for S obtained in [14]. So it is not strange that the (28) should be identical to another nonspectral equation. For completeness, the argument leading to that nonspectral equation is outlined. One notices that (28) is quadratic in terms of z_1, and becomes identity when $z_1 = z_2$ or $z_1 = z_3$. So (28) can be written as

$$(z_1 - z_2)(z_1 - z_3)Q(z_2,z_3) = 0, \tag{31}$$

which means (28) is equivalent to $Q = 0$. To obtain Q, one can simply put $z_1 = 0$ in (28). Doing this, and inverting both sides, another equation is obtained which is quadratic in terms of z_3. Again it is seen that for $z_3 = 0$ and $z_3 = z_2$, this equation becomes identity. So one can write this equation as

$$z_3(z_3 - z_2)\tilde{Q}(z_2) = 0, \tag{32}$$

which is equivalent to $\tilde{Q} = 0$. To find $\tilde{Q}$, one simply writes the coefficient of z_3^2 in the inverted equation. One arrives at an equation containing only z_2. This in turn, can be converted to an expression quadratic in terms of z_2. The coefficients of z_2^0 and z_2^2 of this equation are identities, while the coefficient of z_2 gives

$$b_{2,3}\, b_{1,2}(b_{2,3} + b_{1,2}) = (b_{2,3} + b_{1,2})b_{2,3}\, b_{1,2}, \tag{33}$$

or

$$b_{2,3}[b_{2,3}, b_{1,2}] = [b_{2,3}, b_{1,2}]b_{1,2}, \tag{34}$$

which are the same as equations (47) and (48) in [14], with $b_{1,2} \leftrightarrow b_{2,3}$, as expected.

4 The conditional probability

Assuming that the solvability condition (28), or equivalently (34), is satisfied, One can determine the conditional probability (or the propagator):

$$U(\vec{x};t|\vec{y};0) = \int \frac{d^N p}{(2\pi)^N} e^{-i\vec{p}\cdot\vec{y}} \sum_\sigma A_\sigma\, e^{i\sigma(\vec{p})\cdot\vec{x}}\, e^{t\,E(\vec{p})}, \tag{35}$$

where the integration region for each p_i is $[0, 2\pi]$, and $A_1 = 1$. The singularity in A_σ is removed by setting $p_j \to p_j - i\epsilon$, where the limit $\epsilon \to 0^+$ is meant. Note that as the elements of b are nonnegative and b satisfies (10), the absolute values of the eigenvalues of b are not greater than 1.

For the two-particle sector, there is only one matrix (b) in the expression of U. So, it can be treated as a c number. Using a calculation similar to what has been done in [15] and [14], one arrives at

$$U(\vec{x};t|\vec{y};0) = e^{-2t}\bigg\{ \frac{t^{x_1-y_1}}{(x_1-y_1)!}\frac{t^{x_2-y_2}}{(x_2-y_2)!}$$
$$+ \sum_{l=0}^{\infty} \frac{t^{x_2-y_1}}{(x_2-y_1)!}\frac{t^{x_1-y_2-l}}{(x_1-y_2-l)!}b^l \left[-1 + \frac{(x_2-y_1)b}{t}\right] \bigg\}. \tag{36}$$

To investigate the large-time behavior of the propagator, it is useful to decompose the vector space on which b acts, into two subspaces invariant under the action of b: the first subspace corresponding to eigenvalues with modulus one, the second corresponding to eigenvalues with modulus less than one. This is done introducing two projections Q and R, satisfying

$$Q + R = 1,$$
$$Q\,R = R\,Q = 0,$$
$$[b, Q] = [b, R] = 0. \tag{37}$$

Q projects on the first subspace, and R projects on the second. One can now multiply U by $1 = Q + R$. In the term multiplied by R, one can treat b as a number with modulus less than 1. But if the modulus of b is less than 1, then the integrand in (35) is nonsingular and it is readily seen that for large t, the leading term in the integral is independent of b, and in fact equal to the value obtained with $b = 0$. So

$$U(\vec{x};t|\vec{y};0) = e^{-2t}\bigg\{ \frac{t^{x_1-y_1}}{(x_1-y_1)!}\frac{t^{x_2-y_2}}{(x_2-y_2)!}$$
$$+ \sum_{l=0}^{\infty} \frac{t^{x_2-y_1}}{(x_2-y_1)!}\frac{t^{x_1-y_2-l}}{(x_1-y_2-l)!}b^l$$
$$\times \left[-1 + \frac{(x_2-y_1)b}{t}\right] \bigg\}Q$$
$$+ e^{-2t}\bigg\{ \frac{t^{x_1-y_1}}{(x_1-y_1)!}\frac{t^{x_2-y_2}}{(x_2-y_2)!}$$
$$+ \sum_{l=0}^{\infty} \frac{t^{x_2-y_1}}{(x_2-y_1)!}\frac{t^{x_1-y_2-l}}{(x_1-y_2-l)!}b^l$$
$$\times \left[-1 + \frac{(x_2-y_1)b}{t}\right] \bigg\}R, \tag{38}$$

and for large times,

$$\text{the second term} = \frac{1}{2\pi t}\left\{e^{-\left[(x_1-y_1-t)^2+(x_2-y_2-t)^2\right]/(2t)}\right.$$

$$\left. - e^{-\left[(x_1-y_2-t)^2+(x_2-y_1-t)^2\right]/(2t)}\right\}R, \quad t \to \infty. \quad (39)$$

So at large times, the second term tends to zero faster than t^{-1}, and the leading term in the conditional probability, which is of the order t^{-1}, does not involve the second term.

If the only eigenvalue of b with modulus 1 is 1, then U has a simple behavior for $t \to \infty$. In this case, $bQ = Q$, and one can simplify U to find

$$U(\vec{x};t|\vec{y};0) = e^{-2t}\left[\frac{t^{x_1-y_1}}{(x_1-y_1)!}\frac{t^{x_2-y_2}}{(x_2-y_2)!}\right.$$

$$\left. + \sum_{l=0}^{\infty}\frac{t^{x_2-y_1}}{(x_2-y_1)!}\frac{t^{x_1-y_2-l}}{(x_1-y_2-l)!}\left(-1+\frac{x_2-y_1}{t}\right)\right]Q$$

$$t \to \infty. \quad (40)$$

This is simply the propagator corresponding to a single-species drop-push system, multiplied by Q. This means that for large times, the behavior of the system can in some sense be decomposed to that a single-species drop push model, and an interacting non-diffusive one. If one is not interested in the particle-type, then the characteristics of the drop-push model are recovered (unit drift velocity and a diffusion rate equal to $1 + (1/2) - (1/\pi)$ for large times and in properly-scaled units). This behavior is independent of the reaction rates. If the diffusion is not of interest and only the particle-type is studied, then only the reaction rates (coded in the matrix b) are relevant and the diffusion is unimportant.

5 Dynamics of the particle-type

For simplicity, let's continue with the two-particle sector. The probability that the first particle be of type A_α and the second particle be of type A_β, regardless of their positions, is

$$P^{\alpha\beta}(t) = \sum_{x_1,x_2}{}' P^{\alpha\beta}(x_1,x_2;t), \quad (41)$$

where the primed summation means that the summation is on the physical region ($x_1 < x_2$). Differentiating this, one arrives at

$$\dot{P}^{\alpha\beta}(t) = \sum_{x_1,x_2}{}'\left[P^{\alpha\beta}(x_1-1,x_2;t)\right.$$

$$\left. + P^{\alpha\beta}(x_1,x_2-1;t) - 2P^{\alpha\beta}(x_1,x_2;t)\right],$$

$$= \sum_{x}(b^{\alpha\beta}{}_{\mu\nu} - \delta^{\alpha}{}_{\mu}\delta^{\beta}{}_{\nu})P^{\mu\nu}(x-1,x;t), \quad (42)$$

where in the last inequality, the boundary condition (9) has been used. It is seen that the evolution equation of

the particle-type is not closed; it involves the probability of finding different types of the particles in adjacent sites.

However, the complete conditional probability for large times has the simple form (40). For that form, the summation in (41) is readily done, and one arrives at

$$\sum_{x_1,x_2}{}' U^{\alpha\beta}{}_{\mu\nu}(\vec{x};t|\vec{y};0) = Q^{\alpha\beta}{}_{\mu\nu}, \quad t \to \infty. \quad (43)$$

Here the fact has been used that the multiplier of Q in (4) is simply the propagator of the single-species drop-push model, and its summation on the physical region results in 1. From this, it is seen that the large-time probability of particle-types, depends only on the initial types of the particles, and not on their initial positions. This is of course true, when the only eigenvalues of b with modulus 1 is 1, the condition for (40) to hold. If moreover, this eigenvalue is nondegenerate, then the large-time probability of particle-types is even independent of the initial particle types. In this case, Q would be written like

$$Q^{\alpha\beta}{}_{\mu\nu} = q^{\alpha\beta}s_\mu s_\nu, \quad (44)$$

from which

$$\lim_{t\to\infty} P^{\alpha\beta}(t) = q^{\alpha\beta}. \quad (45)$$

Here, q is the eigenvector of b with eigenvalue 1, normalized as

$$s_\alpha s_\beta q^{\alpha\beta} = 1. \quad (46)$$

It is seen that in this case, the large time probability of the particle types depends only on their interaction when they are adjacent.

6 Concluding remarks

It was seen that a special class of multi-species drop-push models are solvable in the sense of the Bethe-ansatz. The condition corresponding to this solvability, resembles very much to what obtained in [14] for the solvability of multi-species asymmetric exclusion processes. This is not accidental, since the behaviors of the drop-push model and the asymmetric exclusion model on continuum are related to each other: using a Galilean transformation, one can transform a drop-push model in which particles diffuse to the right, to an exclusion model in which particles diffuse to the left [11,12]. So the results obtained in [14], regarding the solvability, with minor modifications can be used here.

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